Some 0/1 polytopes need exponential size extended formulations

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Abstract

We prove that there are 0/1 polytopes $P \subseteq \mathbb{R}^n$ that do not admit a compact LP formulation. More precisely we show that for every n there is a sets $X \subseteq \{0,1\}^n$ such that $\text{conv}(X)$ must have extension complexity at least $2^{n/2\cdot(1-o(1))}$. In other words, every polyhedron Q that can be linearly projected on $\text{conv}(X)$ must have exponentially many facets.

In fact, the same result also applies if $\text{conv}(X)$ is restricted to be a matroid polytope.

Conditioning on $\mathbf{NP} \not\subseteq \mathbf{P_{/poly}}$, our result rules out the existence of any compact formulation for the TSP polytope, even if the formulation may contain arbitrary real numbers.

1 Introduction

Combinatorial optimization deals with finding the best solution out of a finite number of choices $X \subseteq \{0,1\}^n$, e.g. finding the cheapest spanning tree in a graph. If possible one aims of course to design a polynomial time algorithm. However another popular way to study combinatorial problems is to express the convex hull $P = \text{conv}(X)$ by linear inequalities $Ax \leq b$, i.e. describing them as the solutions of a linear program. A drawback of this approach is that in general an exponential number of inequalities is needed. In principle one could use the Ellipsoid method to optimize these systems,

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if at least the separation problem can be solved in polynomial time. But in practice this method is considered to be not applicable. A more satisfactory approach is to allow polynomially many extra variables in order to reduce the number of necessary inequalities to a polynomial. This is called a *compact formulation* $P = \{x \mid \exists y : Ax + Uy \leq b\}$. Such compact formulations exist for example for the spanning tree polytope [Mar91], the parity polytope and the permutahedron (see [Sch03] for an extensive account).

The advantages of such a compact formulation are that (1) one can now optimize any linear function over X in polynomial time; (2) one can solve the problem with a powerful general purpose LP solver, without the need to implement a custom-tailored algorithm.

This naturally leads to the question for which problems such a compact formulation does *not* exist. Yannakakis [Yan91] showed that the TSP polytope P_{TSP} (the convex full of the characteristic vectors of all Hamiltonian cycles in the complete graph on n nodes) does not have a subexponential size *symmetric* formulation. Surprisingly the same result holds true for the matching polytope, though here a complete description of all facets is known due to Edmonds [Edm65] and the problem itself as well as the separation problem are solvable in polynomial time. Kaibel, Pashkovich and Theis [KPT10] demonstrate that symmetric formulations are in some cases more restricted by proving that there is a compact non-symmetric formulation for all $\log n$-size matchings, while symmetric formulations still need size $n^{\Omega(\log n)}$.

However, it remains a fundamental open problem to show that the matching polytope or the TSP polytope do not admit any non-symmetric compact formulation. In fact, it was even an open problem to prove that there *exists* any family of 0/1 polytopes without a compact formulation[1]. In this paper we answer this question affirmatively.

Our idea is based on a counting argument similar to Shannon's theorem [Sha49] (see also [AB09]) for lower bounds on circuit sizes: Let us assume for the sake of contradiction that all n-dimensional 0/1 polytopes have a compact formulation $P = \{x \mid \exists y \geq \mathbf{0} : Ax + Uy = b\}$ of polynomial size $r(n)$. Since there are doubly-exponentially many 0/1 polytopes, there must also be at least that many formulations of size $r(n)$. This would lead to a contradiction under the additional assumption that all coefficients in the system $Ax + Uy = b$ have polynomial encoding length. Unfortunately there is no known result which guarantees that the coefficients of U will even be

[1] This was posed as an open problem by Volker Kaibel on the 1st Cargèse Workshop in Combinatorial Optimization.

rational and already a single real number can contain an infinite amount of information[2] ruling out a simple counting argument.

Our contribution

In our approach, we bypass these difficulties by selecting a linearly independent subsystem of $Ax + Uy = b$ which maximizes the volume of the spanned parallelepiped; then we discretize the entries of U. We thus obtain a subsystem $\bar{A}x + \bar{U}y = \bar{b}$ with the property that $x \in X$ if and only if there is a short certificate y such that $\bar{A}x + \bar{U}y \approx \bar{b}$ for the rounded system. Secondly, all numbers in $\bar{A}, \bar{U}, \bar{b}$ have an encoding length which is bounded by a polynomial in n. In other words, this construction defines an injective map, taking a set X as input and providing $(\bar{A}, \bar{U}, \bar{b})$. Since there are doubly-exponentially many sets $X \subseteq \{0,1\}^n$ and by injectivity, the number of such systems $(\bar{A}, \bar{U}, \bar{b})$ must also be doubly-exponential, which then implies the result.

It is folklore, that if **NP** problems do not all have polynomial size circuits, then the TSP polytope does not admit a compact formulation in which the numbers are rationals with polynomial encoding length. We can argue that the latter condition can be omitted.

2 Related work

A formulation of size $O(n \log n)$ for the permutahedron was provided by Goemans [Goe10]. In fact, [Goe10] also showed that this is tight up to constant factors. The lower bound of [Goe10] is based on the insight that the number of facets of any extension must be at least logarithmic in the number of vertices of the target polytope (which is $n!$ for the permutahedron). The perfect matching polytope for planar graphs and graphs with bounded genus does admit a compact formulation [Bar93, Ger91]. A useful tool to design such formulations is the Theorem of Balas [Bal85, Bal98], which describes the convex hull of the union of polyhedra. For **NP**-hard problems, one can of course not expect the existence of any *exact* compact formulation. Nevertheless, Bienstock [Bie08] gave an approximate formulation of size $n^{O(1/\varepsilon)}$ for the Knapsack polytope. This means, optimizing any linear function over the approximate polytope will give the optimum Knapsack value, up to a

[2]Note that the usual argument that a polytope with rational vertices admits rational inequalities and vice versa does not apply, since both, the vertices and the inequalities of the extension polyhedron might be irrational.

$1+\varepsilon$ factor. For a more detailed literature review, we refer to the surveys of Conforti, Cornuéjols and Zambelli [CCZ10] and of Kaibel [Kai11].

3 Preliminaries

Let $P \subseteq \mathbb{R}^n$ be a polytope with non-redundant inequality representation $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$. An *extension* is a polyhedron $Q \subseteq \mathbb{R}^m$ together with a linear projection $p : \mathbb{R}^m \to \mathbb{R}^n$ such that $p(Q) = P$. An *extended formulation* is a description of Q with linear inequalities and equations $Q = \{z \in \mathbb{R}^m \mid Cz \leq c,\ Dz = d\}$ (together with p). The *size* of the extended formulation is the number of inequalities in the description, i.e. the number of rows in C. We do not need to account for the number of equations, since they can always be eliminated. Now we can define the *extension complexity* $\mathrm{xc}(P)$ as the smallest size of any extended formulation (see [Kai11] for more details).

Let $X = \{x_1, \ldots, x_v\} \subseteq P$ be the *vertices* (or *extreme points*) of P and let f be the number of inequalities in the description $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$. Then the *slack-matrix* $S \in \mathbb{R}^{f \times v}$ of P is defined by $S_{ij} = b_i - A_i x_j$. Recall that the *rank* of a matrix S is the smallest r such that one can factor $S = UV$, where U is a matrix with r columns and V is a matrix with r rows. A notion which is very important for studying extended formulations is the *non-negative rank* of a matrix:

$$\mathrm{rk}_+(S) = \min\{r \mid \exists U \in \mathbb{R}^{f \times r}_{\geq 0}, V \in \mathbb{R}^{r \times v}_{\geq 0} : S = UV\}$$

Note that given a matrix $A \subseteq \mathbb{Q}^{m \times n}_{\geq}$, deciding whether $\mathrm{rk}(A) = \mathrm{rk}_+(A)$ is **NP**-hard [Vav09]. A basic theorem concerning extended formulations, is the insight of Yannakakis, that the non-negative factorization of the slack-matrix with minimum r gives the smallest extension:

Theorem 1 (Yannakakis [Yan91])**.** *Let P be a polytope with vertices $X = \{x_1, \ldots, x_v\}$, non-redundant inequality description $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ and corresponding slack matrix S. Then $\mathrm{xc}(P) = \mathrm{rk}_+(S)$. Moreover, for any factorization $S = UV$ with $U, V \geq \mathbf{0}$ one can write $P = \{x \in \mathbb{R}^n \mid \exists y \geq \mathbf{0} : Ax + Uy = b\}$ and for every $x_j \in X$ one has $Ax_j + U \cdot V^j = b$.*

In other words: Given a polytope $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$, the smallest extension can be found by factoring the slack matrix S into non-negative factors U and V with minimum number of columns/rows. Then the smallest extended formulation comprises of $Q = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^{\mathrm{xc}(P)} \mid Ax + Uy =$

$b,\ y \geq \mathbf{0}\}$ together with the *projection on the x-variables* $\mathrm{proj}_x(Q) = \{x \in \mathbb{R}^n \mid \exists y : (x,y) \in Q\}$. While for a polytope P, the inequality description $Ax \leq b$ is not unique, Theorem 1 implies that the non-negative rank is the same for all these descriptions.

For any matrix A, we denote its ith row by A_i and the ith column by A^i. For linearly independent vectors $w_1, \ldots, w_k \in \mathbb{R}^n$, we define $\mathrm{vol}(w_1, \ldots, w_k)$ as the k-dimensional volume of the parallelepiped, spanned by $w_1, \ldots, w_k$. Hence for $k = n$ one has $\mathrm{vol}(w_1, \ldots, w_k) = |\det(B)|$ where B is a matrix, having $w_1, \ldots, w_k$ as column vectors in an arbitrary order. Note that for any vector $w \in \mathit{span}(w_1, \ldots, w_k)$, there are unique coefficients $\lambda \in \mathbb{R}^k$ such that $w = \sum_{i=1}^k \lambda_i w_i$ and by Cramer's rule

$$|\lambda_i| = \frac{\mathrm{vol}(w_1, \ldots, w_{i-1}, w, w_{i+1}, \ldots, w_k)}{\mathrm{vol}(w_1, \ldots, w_k)}.$$

For $q \in \mathbb{R}$, let $q\mathbb{Z}_{\geq 0} = \{0, q, 2q, \ldots\}$ denote all non-negative integer multiples of q.

4 A lower bound for general 0/1 polytopes

In the following we fix a set $X \subseteq \{0,1\}^n$. It is well known, that one can choose a matrix A and a vector b with integral entries such that $P = \{x \in \mathbb{R}^n \mid Ax \leq b\} = \mathrm{conv}(X)$, while the absolute values of any entry in A and b are bounded by $\Delta := \Delta(n) := (\sqrt{n+1})^{n+1} \leq 2^{n \log(2n)}$ (see e.g. Cor. 26 in [Zie00]). Let S be the corresponding slack-matrix, then S is non-negative by definition and integral, since A, b and all vertices are integral. More precisely $S_{ij} = b_j - A_i x_j \in \{0, \ldots, (n+1)\Delta\}$. Let $S = UV$ be any non-negative factorization, i.e. $U \in \mathbb{R}_{\geq 0}^{f \times r}$ and $V \in \mathbb{R}_{\geq 0}^{r \times v}$. As already argued above, we cannot make any assumption on the rationality/encoding length of the coefficients of U and V. But what we can do is to bound their absolute values.

Observe that if we simultaneously scale a column ℓ of U by $\lambda > 0$ and row ℓ of V by $\frac{1}{\lambda}$, then the matrix product UV stays invariant. Thus we may scale the rows and columns such that $\|U^\ell\|_\infty = \|V_\ell\|_\infty$ (if $U^\ell = \mathbf{0}$, then we can just set $V_\ell := \mathbf{0}$ as well). We call such pairs of matrices *normalized*.

Lemma 2. *For normalized matrices, one has $\|U\|_\infty \leq \Delta$ and $\|V\|_\infty \leq \Delta$.*

Proof. Assume for the sake of contradiction that $U_{i\ell} > \Delta$. Thus $\|V_\ell\|_\infty > \Delta$, hence there must be an entry $V_{\ell j} > \Delta$. Then $S_{ij} = U_i \cdot V^j \geq U_{i\ell} \cdot V_{\ell j} > \Delta^2 \geq (n+1)\Delta$, which is a contradiction. □

Recalling Theorem 1, we can write $\text{conv}(X) = \{x \in \mathbb{R}^n \mid \exists y \in \mathbb{R}^{\text{xc}(\text{conv}(X))}_{\geq 0} : Ax + Uy = b\}$. Our main technical ingredient is to select a linear independent subsystem $\bar{A}x + \bar{U}y = \bar{b}$ of $Ax + Uy = b$ such that the entries of $\bar{U}$ can be rounded to rational numbers with small encoding length and still $x \in X$ iff $\bar{A}x + \bar{U}y \approx \bar{b}$ for some y.

Theorem 3. *For any non-empty $X \subseteq \{0,1\}^n$, there are matrices $\bar{A} \in \mathbb{Z}^{(n+r)\times n}, \bar{U} \in (\frac{1}{4r(n+r)\Delta}\mathbb{Z}_{\geq 0})^{(n+r)\times r}$ and a vector $\bar{b} \in \mathbb{Z}^{n+r}$ with $\|\bar{A}\|_\infty, \|\bar{b}\|_\infty, \|\bar{U}\|_\infty \leq \Delta$ such that*

$$X = \left\{ x \in \{0,1\}^n \mid \exists y \in [0,\Delta]^r : \|\bar{A}x + \bar{U}y - \bar{b}\|_\infty \leq \frac{1}{4(n+r)} \right\}$$

Here is $r := xc(conv(X))$ and $\Delta := \Delta(n) := (\sqrt{n+1})^{n+1}$.

Proof. Let $X = \{x_1, \ldots, x_v\}$ and let $Ax \leq b$ with $A \in \mathbb{Z}^{f\times n}$ and $b \in \mathbb{Z}^f$ be a non-redundant description of $\text{conv}(X)$ with $\|A\|_\infty, \|b\|_\infty \leq \Delta$. Furthermore let $S \in \mathbb{Z}^{f\times|X|}_{\geq 0}$ be the corresponding slack matrix. By Yannakakis' Theorem 1, we can write $P = \text{conv}(X) = \{x \in \mathbb{R}^n \mid \exists y \in \mathbb{R}^r : Ax + Uy = b,\ y \geq \mathbf{0}\}$ where U, V are the non-negative factorization of the slack-matrix, i.e. $S = UV$. By Lemma 2 we may assume that $\|U\|_\infty, \|V\|_\infty \leq \Delta$. Let $W = \text{span}(\{(A_i, U_i) \mid i = 1, \ldots, f\})$ be the span of the constraint matrix of the system $Ax + Uy = b$ and let $k = \dim(W)$ be its dimension. Choose $I \subseteq \{1, \ldots, f\}$ of size $|I| = k$ such that $\text{vol}(\{(A_i, U_i) \mid i \in I\})$ is maximized. Recall that U_I is the matrix U, restricted to the rows in I. Let U'_I be the matrix U_I where coefficients are rounded down to the nearest multiple of $\frac{1}{4r(n+r)\Delta}$. Our choice will be $\bar{A} := A_I, \bar{U} := U'_I, \bar{b} := b_I$, hence it remains to show that

$$X \overset{!}{=} \left\{ x \in \{0,1\}^n \mid \exists y \in [0,\Delta]^r : \|A_I x + U'_I y - b_I\|_\infty \leq \frac{1}{4(n+r)} \right\} =: Y$$

Claim. $X \subseteq Y$.

Proof of claim. Consider a vector $x_j \in X$. Using Yannakakis' Theorem 1, we can simply choose $y := V^j \geq \mathbf{0}$ and have $Ax_j + U \cdot y = b$. Due to normalization, $\|y\|_\infty \leq \|V\|_\infty \leq \Delta$. Note that $\|U - U'\|_\infty \leq \frac{1}{4r(n+r)\Delta}$. By

the triangle inequality

$$\begin{aligned}\|A_I x_j + U'_I y - b_I\|_\infty &\leq \|\underbrace{A_I x_j + U_I y - b_I}_{=\mathbf{0}} + (U'_I - U_I)y\|_\infty \\ &\leq r \cdot \underbrace{\|U'_I - U_I\|_\infty}_{\leq \frac{1}{4r(n+r)\Delta}} \cdot \underbrace{\|y\|_\infty}_{\leq \Delta} \leq \frac{1}{4(n+r)}\end{aligned}$$

Thus $x_j \in Y$. $\Diamond$

Claim. $X \supseteq Y$.

Proof of claim. We show that for $x \in \{0,1\}^n$ with $x \notin X$ one has $x \notin Y$. Since $x \notin X$, there must be a row ℓ with $A_\ell x > b_\ell$. Since A, b and x are integral, one even has $A_\ell x \geq b_\ell + 1$. Unfortunately ℓ is in general not among the selected constraints I. But there are unique coefficients $\lambda \in \mathbb{R}^k$ such that we can express constraint $A_\ell x + U_\ell y = b_\ell$ as a linear combination of those in I, i.e.

$$\left(A_\ell, U_\ell\right) = \sum_{i \in I} \lambda_i \left(A_i, U_i\right).$$

Note that automatically we have $\sum_{i \in I} \lambda_i b_i = b_\ell$, since otherwise the system $Ax + Uy = b$ could not have any solution (x, y) at all and $X = \emptyset$. The next step is to bound the coefficients λ_i. Here we recall that by Cramer's rule

$$|\lambda_i| = \frac{\mathrm{vol}\big(\big\{(A_{i'}, U_{i'}) \mid i' \in I \backslash \{i\} \cup \{\ell\}\big\}\big)}{\mathrm{vol}\big(\big\{(A_{i'}, U_{i'}) \mid i' \in I\big\}\big)} \leq 1$$

since we picked I such that $\mathrm{vol}(\{(A_{i'}, U_{i'}) \mid i' \in I\})$ is maximized. Fix an arbitrary $y \in [0, \Delta]^r$, then

$$\begin{aligned}1 \leq |\underbrace{A_\ell x - b_\ell}_{\geq 1} + \underbrace{U_\ell y}_{\geq 0}| &= \Big|\sum_{i \in I} \lambda_i (A_i x - b_i + U_i y)\Big| \qquad (1) \\ &\leq \sum_{i \in I} \underbrace{|\lambda_i|}_{\leq 1} \cdot |A_i x - b_i + U_i y| \\ &\leq (n + r) \cdot \|A_I x - b_I + U_I y\|_\infty\end{aligned}$$

using the triangle inequality and the fact that $|I| \leq n + r$. Again making

use of the triangle inequality yields

$$\begin{aligned} \|A_I x - b_I + U_I y\|_\infty &= \|A_I x - b_I + U_I' y + (U_I - U_I')y\|_\infty \\ &\leq \|A_I x - b_I + U_I' y\|_\infty + r \cdot \underbrace{\|U_I - U_I'\|_\infty}_{\leq \frac{1}{4r(n+r)\Delta}} \cdot \underbrace{\|y\|_\infty}_{\leq \Delta} \\ &\leq \|A_I x - b_I + U_I' y\|_\infty + \frac{1}{4(n+r)} \end{aligned} \tag{2}$$

Combining (1) and (2) gives $\|A_I x - b_I + U_I' y\|_\infty \geq \frac{1}{n+r} - \frac{1}{4(n+r)} \geq \frac{1}{2(n+r)}$ and consequently $x \notin Y$. ♢

The assertion of the Theorem follows. Note that by padding empty rows, we can ensure that $\bar{A}, \bar{U}, \bar{b}$ have exactly $n + r$ rows. □

Theorem 4. *For any $n \in \mathbb{N}$, there exists a set $X \subseteq \{0,1\}^n$ such that $xc(conv(X)) \geq \Omega(2^{n/2}/\sqrt{n \log(2n)})$.*

Proof. Let $R := R(n)$ be the maximum value of $\mathrm{xc}(\mathrm{conv}(X))$ over all $X \subseteq \{0,1\}^n$. In the following, we use that $R \leq 2^n$ (otherwise, there is nothing to show). The construction in Theorem 3 implicitly defines a function Φ which maps a set X to a system $(\bar{A}, \bar{U}, \bar{b})$[3]. The important observation is that due to Theorem 3, for a given system $(\bar{A}, \bar{U}, \bar{b})$, one can reconstruct the corresponding set X. In other words, the function Φ is injective. In fact, adding zero rows and columns to those matrices does not change the claim, hence we may assume that $\bar{A}$ is an $(n+R) \times n$ matrix and $\bar{U}$ is an $(n+R) \times R$ matrix. Every entry in $\bar{U}$ has absolute value at most Δ and is a multiple of $\frac{1}{4r(n+r)\Delta}$ for some $r \in \{1, \ldots, R\}$. In other words, the domain for each entry contains at most $\sum_{r=1}^{R} 2 \cdot 4r(n+r)\Delta \cdot \Delta \leq 8R^2(n+R^2)\Delta \leq 16\Delta^5$ many possible values (here we use the generous estimates $R \leq 2^n \leq \Delta$ and $n \leq \Delta$). By injectivity of Φ, the number of sets X (which is $2^{2^n} - 1$) cannot be larger than the number of systems $(\bar{A}, \bar{U}, \bar{b})$. Thus

$$2^{2^n} - 1 \leq (16\Delta^5)^{(n+R+1)\cdot(n+R)} \leq 2^{C(n^4 + n\log(2n) \cdot R^2)}$$

for some constant $C > 0$. Hence $R \geq C' \cdot 2^{n/2}/\sqrt{n \log(2n)}$ for some $C' > 0$. □

[3]The initial system $Ax \leq b$ describing $\mathrm{conv}(X)$ might not be unique, as well as index set I. For Φ to be well defined one can make an arbitrary canonical choice, like choosing $Ax \leq b$ and I lexicographical minimal.

5 A lower bound for matroid polytopes

The main drawback of our result is that it does not rule out compact formulations for any explicitly known polytope. However, we can extend the result to matroid polytopes. Recall that a pair $([n], \mathcal{I})$ is called a *matroid* with *ground set* $[n] = \{1, \dots, n\}$ and *independent sets* $\mathcal{I} \subseteq 2^{[n]}$, if (I) $I \in \mathcal{I}, J \subseteq I \Rightarrow J \in \mathcal{I}$ and (II) for all $I, J \in \mathcal{I}$ with $|I| < |J|$ there is a $z \in J \backslash I$ with $I + z \in \mathcal{I}$. Note that all non-trivial facet-defining inequalities for $\text{conv}(\chi(\mathcal{I}))$ are of the form $\sum_{i \in S} x_i \leq r_{\mathcal{I}}(S)$ with $S \subseteq [n]$, where $r_{\mathcal{I}}$ denotes the *rank function* of the matroid ($\chi(\mathcal{I})$ denotes the set of characteristic vectors of $\mathcal{I}$). Secondly, any linear objective function can be optimized over $\text{conv}(\chi(\mathcal{I}))$ using the greedy algorithm, which involves calling a membership oracle a polynomial number of times. See e.g. the textbook of Schrijver [Sch03] for more details.

Nevertheless, it is well known that the number of matroids with ground set $\{1, \dots, n\}$ is at least $2^{\binom{n}{\lfloor n/2 \rfloor}/(2n)} \geq 2^{2^n/(10n^{3/2})}$ for n large enough [Duk03]. In other words, there are doubly-exponentially many matroids. Using the same proof as for Theorem 4 we obtain:

Corollary 5. *There exists a family $M_n = (\{1, \dots, n\}, \mathcal{I}_n)$ of matroids such that $\text{xc}(\text{conv}(\chi(\mathcal{I}_n))) = \Omega(2^{n/2}/(n^{5/4} \log(2n)))$.*

6 Approximating $0/1$ polytopes

In this section, we want to extend the result of Theorem 3 such that any $0/1$ polytope P can be arbitrarily well approximated as a projection of a polytope Q with $O(n + \text{xc}(P))$ facets but still small encoding length. See Figure 6 for an illustration. In the following, for any $\varepsilon > 0$, let $P + \varepsilon = \{x + z \in \mathbb{R}^n \mid x \in P, \|z\|_2 \leq \varepsilon\}$.

Theorem 6. *For any non-empty $0/1$ polytope $P = \text{conv}(X)$ ($X \subseteq \{0,1\}^n$) and any $\varepsilon > 0$, there exists a polytope $Q = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^{\text{xc}(P)} \mid Bx + Cy \leq d\}$ such that $B \in \mathbb{Q}^{(4\text{xc}(P)+2n) \times n}, C \in \mathbb{Q}^{(4\text{xc}(P)+2n) \times \text{xc}(P)}$ and $b \in \mathbb{Q}^{4\text{xc}(P)+2n}$ have encoding length $\text{poly}(n, \text{xc}(P), \log(\frac{1}{\varepsilon}))$ and $P \subseteq \text{proj}_x(Q) \subseteq P + \varepsilon$.*

Furthermore for any objective function $c \in \mathbb{R}^n$, $\max\{c^T x \mid x \in \text{proj}_x(Q)\} - \max\{c^T x \mid x \in P\} \leq \varepsilon \cdot \|c\|_2$.

Proof. Again let $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ be a non-redundant inequality description of P such that A and b have entries from $\{-\Delta, \dots, \Delta\}$. Abbreviate $r := \text{xc}(P)$. We again apply Theorem 3 to obtain a system A_I, U'_I, b_I.

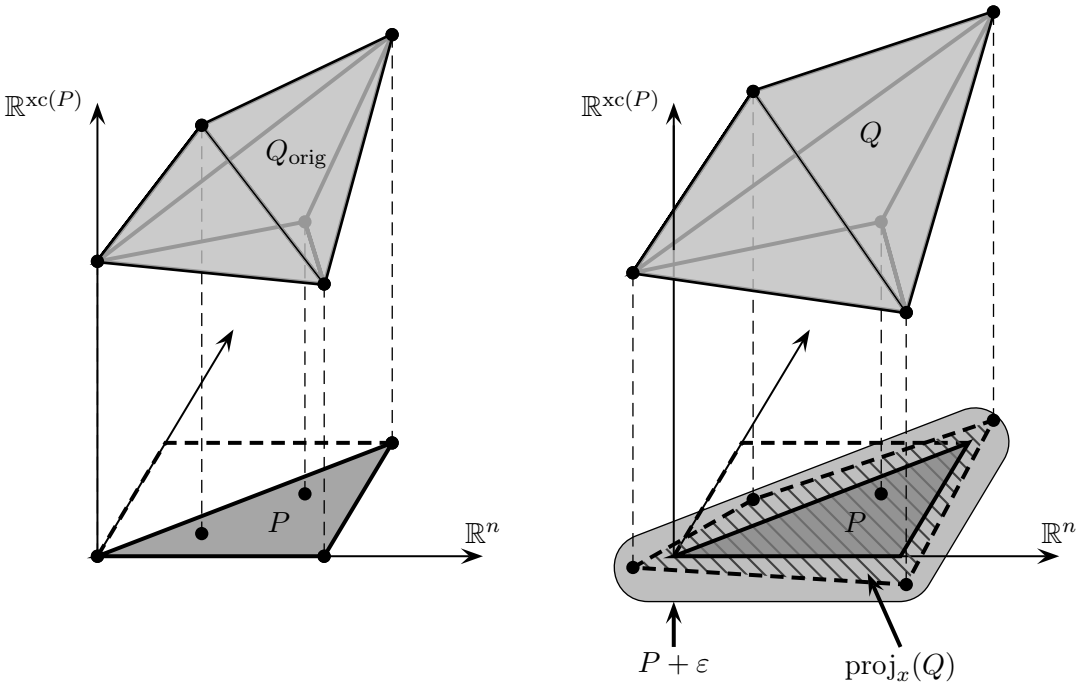

Figure 1: Visualization of Theorem 6.

But this time, we round the entries in the matrix U_I down to the nearest multiple of $\frac{\delta}{4r(n+r)\Delta}$ (instead of $\frac{1}{4r(n+r)\Delta}$), for $\delta := \min\{\frac{1}{2(n\Delta)^{2n+2}}, \frac{\varepsilon}{n\cdot(n\Delta)^n}\}$. We choose

$$Q := \left\{(x,y) \mid \|A_I x + U_I' y - b_I\|_\infty \leq \frac{\delta}{4(n+r)}, y \in [0,\Delta]^r\right\}$$

Note that Q is in fact a polytope which can be written in the form $Q = \{(x,y) \mid Bx + Cy \leq d\}$ such that B, C, d are of the claimed format. Furthermore the encoding length of B, C, d is polynomial in n, $\mathrm{xc}(P)$ and $\log(1/\varepsilon)$[4]. In the remaining proof we show that $P \subseteq \mathrm{proj}_x(Q) \subseteq \{x \in \mathbb{R}^n \mid Ax \leq b + \delta\mathbf{1}\} \subseteq P + \varepsilon$.

Claim. *$P \subseteq proj_x(Q)$.*

Proof of claim. As in Theorem 3, for any vertex $x_j \in P$, one has $(x_j, V^j) \in Q$ (since $\|A_I x_j + U_I' V^j - b_I\|_\infty \leq r \cdot \|U_I' - U_I\|_\infty \cdot \|V^j\|_\infty \leq \frac{\delta}{4(n+r)}$). Consequently $P \subseteq \mathrm{proj}_x(Q)$. ◊

[4]This follows from the fact that all coefficients in B, C, d are products of $n, \mathrm{xc}(P), \delta, \varepsilon, \Delta$ (or their reciprocals) and $\log(\Delta) \leq O(n \cdot \log n), \log(1/\delta) \leq \log(1/\varepsilon) + O(n^2 \log n)$.

Claim. $proj_x(Q) \subseteq \{x \in \mathbb{R}^n \mid Ax \leq b + \delta\mathbf{1}\}$.

Proof of claim. Suppose for the sake of contradiction, that there is an $x^* \in \text{proj}_x(Q)$ such that for some ℓ one has $A_\ell x^* > b_\ell + \delta$. Revisiting again Inequalities (1) and (2), we see that for any $y \in [0, \Delta]^r$ now

$$\begin{aligned}
\delta &\overset{(1)}{\leq} (n+r) \cdot \|A_I x^* - b_I + U_I\|_\infty \\
&\overset{(2)}{\leq} (n+r) \cdot \Big(\|A_I x^* - b_I + U_I' y\|_\infty + r \cdot \underbrace{\|U_I - U_I'\|_\infty}_{\leq \delta/(4r(n+r)\Delta)} \cdot \underbrace{\|y\|_\infty}_{\leq \Delta} \Big) \\
&\leq (n+r) \cdot \|A_I x^* - b_I + U_I' y\|_\infty + \frac{\delta}{4}
\end{aligned}$$

which implies that $\|A_I x^* - b_I + U_I' y\|_\infty \geq \frac{\delta}{n+r} - \frac{\delta}{4(n+r)} > \frac{\delta}{4r(n+r)}$ and consequently $x^* \notin \text{proj}_x(Q)$. This is a contradiction. ◊

Claim. $\{x \in \mathbb{R}^n \mid Ax \leq b + \delta\mathbf{1}\} \subseteq P + \varepsilon$.

Proof of claim. It suffices to prove that every vertex x^* of $\{x \mid Ax \leq b + \delta\mathbf{1}\}$ has a distance of at most ε to P. There is a subsystem $A_J x \leq b_J + \delta\mathbf{1}$ of n constraints such that x^* is the unique solution of $A_J x = b_J + \delta\mathbf{1}$ or in other words $x^* = A_J^{-1}(b + \delta\mathbf{1})$. Since A has integral entries with absolute value at most Δ, we know that we can write $A_J^{-1} = (\frac{\alpha_{ij}}{\beta})_{i,j}$ with $\alpha_{ij}, \beta \in \{-(n\Delta)^n, \ldots, (n\Delta)^n\}$[5].

Let us assume for the sake of contradiction that J was not a feasible basis for P, i.e. $A(A_J^{-1} b_J) \not\leq b$. Well, then there is an index i with $A_i(A_J^{-1} b_J) > b_i$. In fact, even $A_i(A_J^{-1} b_J) \geq b_i + \frac{1}{\beta}$. But since we picked δ small enough, $|A_i x^* - A_i(A_J^{-1} b_J)| = |A_i A_J^{-1} \delta\mathbf{1}| \leq n^2 \cdot \Delta \cdot (n\Delta)^n \delta < \frac{1}{(n\Delta)^n} \leq \frac{1}{\beta}$, which is a contradiction.

Hence we may assume that J is indeed a feasible basis for P and we can bound the distance of x^* to P by the distance that the basic solution corresponding to basis J "moved" by shifting the hyperplanes by δ (see Figure 2):

$$\|x^* - A_J^{-1} b_J\|_2 = \|A_J^{-1}(b_j + \delta\mathbf{1}) - A_J^{-1} b_J\|_2 = \|A_j^{-1}\delta\mathbf{1}\|_2 \leq n \cdot \delta \cdot (\Delta n)^n \leq \varepsilon.$$

Here we again used our choice of δ. ◊

Combining the proven claims yields $P \subseteq \text{proj}_x(Q) \subseteq P + \varepsilon$. □

[5] By Cramer's rule, every entry (i,j) of the inverse of an $n \times n$ matrix M can be written as $\pm\frac{\det(M')}{\det(M)}$ for some submatrix M' of M. By the Hadamard bound, $|\det(M)| \leq \prod_{i=1}^n \|M^i\|_2 \leq (n\|M\|_\infty)^n$.

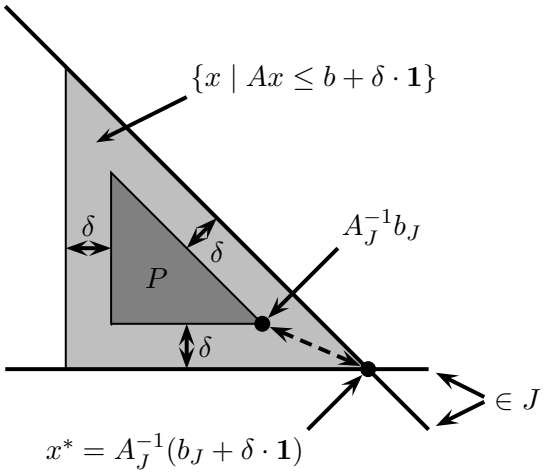

Figure 2: We bound the distance of x^* to P by the distance to $A_J^{-1} b_J$ (see dashed line).

7 Complexity theory considerations

The set of problems that admit compact formulations induce a non-uniform complexity class in a natural way. In the following, we want to briefly discuss, how this class relates to other, well studied classes. For an up-to-date introduction into the topic of complexity theory, we recommend the textbook of [AB09]. Recall that $\{0,1\}^* = \bigcup_{n \geq 0} \{0,1\}^n$ is the set of all binary strings. By a slight abuse of notation we consider a 0/1 string of length n also as a binary vector of dimension n.

Definition 1. *Let* **CF** *be the set of languages $L \subseteq \{0,1\}^*$ for which there exists a polynomial p such that for all $n \in \mathbb{N}$ there exist $A \in \mathbb{R}^{p(n) \times n}, B \in \mathbb{R}^{p(n) \times p(n)}, b \in \mathbb{R}^{p(n)}$ such that*

$$conv(\{x \in L : |x| = n\}) = \{x \in \mathbb{R}^n \mid \exists y \in \mathbb{R}^{p(n)} : Ax + By \leq b\}.$$

By $\mathbf{CF}^{enc} \subseteq \mathbf{CF}$ *we denote the subclass of languages, for which there exist integral matrices A, B and vectors b such that* $\log(\max\{\|A\|_\infty, \|B\|_\infty, \|b\|_\infty\}) \leq p(n)$.

Since any LP of polynomial size and encoding length can be solved in polynomial time, it is rather obvious that $\mathbf{CF}^{\text{enc}} \subseteq \mathbf{P}_{/\mathbf{poly}}$ (see also the remark of Yannakakis [Yan91]). However, Theorem 3 also provides a slightly stronger claim:

Theorem 7. $\mathbf{CF} \subseteq \mathbf{P}_{/\mathbf{poly}}$.

Proof. Let $L \in \mathbf{CF}$ and $X = L \cap \{0,1\}^n$ for some $n \in \mathbb{N}$ and let $r := \mathrm{xc}(\mathrm{conv}(X))$. Recall that r must be polynomial in n. It suffices to provide a Turing machine that takes polynomial advice (see [AB09]). Our advice for all input strings x of length n consists in the matrices $\bar{A}, \bar{U}, \bar{b}$ provided by Theorem 3. Note that their encoding length is bounded by a polynomial in n and r. To verify whether $x \in X$, we simply test whether the following polynomial size linear system has a solution y:

$$\begin{aligned} -\frac{1}{4r(n+r)} \leq \bar{A}x + \bar{U}y - \bar{b} &\leq \frac{1}{4r(n+r)} \\ 0 \leq y_j &\leq \Delta \quad \forall j = 1, \ldots, r \end{aligned}$$

This can be done in polynomial time [Kha79]. □

We make the following conjecture:

Conjecture 8. $\mathbf{CF}^{enc} = \mathbf{CF}$.

One of the most popular polytopes in the literature is the *TSP polytope* (see e.g. [Yan91, BS96]), hence we want to discuss how it relates to the class **CF**. Let K_n be the complete undirected graph on n nodes. We define a language

$$\mathtt{TSP} = \bigcup_{n \in \mathbb{N}} \{\chi(C) \in \mathbb{R}^{\binom{n}{2}} \mid C \subseteq E_n \text{ is Hamiltonian cycle in } K_n = ([n], E_n)\}$$

(here $\chi(C)$ denotes the characteristic vector of C). Again it is obvious that $\mathbf{NP} \not\subseteq \mathbf{P}_{/\mathbf{poly}} \Rightarrow \mathtt{TSP} \notin \mathbf{CF}^{\mathrm{enc}}$, but also here we can show a slightly stronger claim:

Theorem 9. $\mathbf{NP} \not\subseteq \mathbf{P}_{/\mathbf{poly}} \Rightarrow \mathtt{TSP} \notin \mathbf{CF}$. *In other words, unless* **NP** *problems do not all have polynomial size circuits, the TSP polytope does not have a compact formulation, even if arbitrary real numbers are allowed.*

Proof. Suppose for the sake of contradiction that $\mathtt{TSP} \in \mathbf{CF}$. By **NP**-hardness of the *Hamiltonian Cycle problem* [GJ79], given a cost vector $c \in \{1,2\}^{\binom{n}{2}}$ it is **NP**-hard to decide, whether there is an $x \in \mathtt{TSP}$ with $c^T x \leq n$. Consider the Turing machine (taking polynomial advice), which optimizes c over the polytope Q from Theorem 6 for $\varepsilon := \frac{1}{2n}$ and let x^* be an optimum fractional solution. If there is an $x \in \mathtt{TSP}$ with $c^T x \leq n$, then $c^T x^* \leq n$. Otherwise, $c^T x^* \geq (n+1) - \varepsilon \|c\|_2 > n$. Hence the Turing machine decides an **NP**-hard problem, which implies the claim. □

Note that $\texttt{TSP} \in \mathbf{P}_{/\mathbf{poly}}$, since *testing* whether x is the characteristic vector of a Hamiltonian cycle is easy. Just *optimizing* over all those vectors is difficult.

We should not introduce a new complexity class **CF**, without relating it to already known ones. We saw already that $\mathbf{CF} \subseteq \mathbf{P}_{/\mathbf{poly}}$, so what about other non-uniform complexity classes within $\mathbf{P}_{/\mathbf{poly}}$? Certainly the most studied of those classes is $\mathbf{AC}^0$, which is the set of languages for which there are circuits with bounded depth and unbounded fan-in.

Recall that PARITY is the set of all $x \in \{0,1\}^*$ such $\|x\|_1$ is odd. Then PARITY admits a compact formulation (with small integral coefficients; see e.g. [CCZ10]), thus $\texttt{PARITY} \in \mathbf{CF}^{\text{enc}}$. In a seminal result, Furst, Saxe and Sipser [FSS84] showed that $\texttt{PARITY} \notin \mathbf{AC}^0$ and hence $\mathbf{CF} \not\subseteq \mathbf{AC}^0$ (in fact, even $\mathbf{CF}^{\text{enc}} \not\subseteq \mathbf{AC}^0$). On the other hand, under widely believed assumptions also the reverse is true:

Theorem 10. $\mathbf{NP} \not\subseteq \mathbf{P}_{/\mathbf{poly}} \Rightarrow \mathbf{AC}^0 \not\subseteq \mathbf{CF}$.

Proof. We need to exhibit a problem, which can be solved by constant depth circuits, but is likely not to be in **CF**. Consider the complete tripartite graph $G_n = ([n]^3, E_n)$, i.e. for any distinct $i,j,k \in [n]$, one has a triple $e = \{i,j,k\} \in E_n$. We say that a subset $E' \subseteq E_n$ is a *(3-dimensional) matching* if all triples in E' are disjoint. Define

$$\texttt{3DM} = \bigcup_{n \geq 1} \{\chi(E') \mid E' \subseteq E_n \text{ is matching}\}$$

Given a cost vector $c \in \{0,1\}^{E_n}$, it is **NP**-hard to decide, whether there is an $x \in \texttt{3DM}$ with $c^T x = n$ [GJ79] (i.e. whether there is a perfect 3-dimensional matching contained in $\{e \in E \mid c_e = 1\}$). Within the same line of arguments as in Theorem 9 one has $\texttt{3DM} \notin \mathbf{CF}$ unless $\mathbf{NP} \subseteq \mathbf{P}_{/\mathbf{poly}}$. Finally it is not difficult to see that

$$\bigwedge_{e,e' \in E : 1 \leq |e \cap e'| \leq 2} (\neg x_e \vee \neg x_{e'})$$

is a polynomial size, constant depth formula for 3DM, thus $\texttt{3DM} \in \mathbf{AC}^0$. □

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